ALGEBRAIC STRUCTURES ON COHOMOLOGY OF CONFIGURATION SPACES OF MANIFOLDS WITH FLOWS

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ABSTRACT. Let $\operatorname{PConf}^n M$ be the configuration space of ordered n-tuples of distinct points on a smooth manifold M admitting a nowhere-vanishing vector field. We show that the i^{th} cohomology group with coefficients in a field $H^i(\operatorname{PConf}^n M, k)$ is an \mathcal{N} -module, where \mathcal{N} is the category of noncommutative finite sets introduced by Pirashvili and Richter [PR02]. Studying the representation theory of \mathcal{N} , we obtain new polynomiality results for the cohomology groups $H^i(\operatorname{PConf}^n M, k)$. In the case of unordered configuration space $\operatorname{Conf}^n M = (\operatorname{PConf}^n M)/S_n$ and rational coefficients, we show that cohomology dimension is nondecreasing: $\dim H^i(\operatorname{Conf}^n M, \mathbb{Q}) \leq \dim H^i(\operatorname{Conf}^{n+1} M, \mathbb{Q})$.

If M is a smooth manifold, we denote by $\operatorname{Conf}^n M$ the configuration space parametrizing unordered n-tuples (p_1, \ldots, p_n) of distinct points in M. By $\operatorname{PConf}^n M$, the pure or ordered configuration space, we mean the space parametrizing ordered n-tuples of distinct points; this space thus carries a free action of S_n , the quotient by which is $\operatorname{Conf}^n M$.

There is a large existing literature concerning homological stability for configuration spaces. A foundational result of Arnol'd gives an explicit description of the cohomology of $PConf^n \mathbb{R}^2$ [Arn69]. Cohen provides a similar result for $PConf^n \mathbb{R}^m$ [Coh88]. The case of open manifolds M goes back to work of McDuff and Segal [McD75] [Seg74], who study the unordered case. Later work of Church, Farb, Nagpal, and the first author [CEFN12] provides a detailed account of symmetric group characters in cohomology in the ordered case. They show that these characters are polynomial in a certain sense. Their work relies on an $\mathcal{FI}\#$ -structure on $PConf^n M$ that introduces points "from infinity."

For closed manifolds, there may be topological obstructions to introducing points. Nevertheless, in [Chu12], Church proves that characters appearing in the cohomology of configuration spaces with rational coefficients are eventually polynomial using "representation stability" phenomena in the spirit of [CF13]. A subsequent proof of eventual polynomiality using the \mathcal{FI}^{op} -structure on PConfⁿ M appears in [CEF15]. More traditional proofs have now appeared: see [RW13] and [Knu14].

Another way to add points to a configuration on a closed manifold appears in recent work of Cantero-Palmer [CP14] and Berrick, Cohen, Wong, and Wu [BCWW06]; the idea is to introduce new points infinitesimally near old points. These constructions rely on the existence of a nowhere-vanishing vector field on M.

The main contribution of the present paper is to use this geometric information associated with M – namely, the existence of a nowhere-vanishing vector field, or multiple such fields – to improve eventual polynomiality to immediate polynomiality for some closed manifolds, via consideration of the natural module structures enjoyed by the cohomology of the configuration spaces of such a manifold.

We recall the **category of non-commutative finite sets** \mathcal{N} introduced by Pirashvili and Richter in [PR02]. The objects of \mathcal{N} are finite sets; the morphisms are finite set maps

together with a linear ordering on each fiber. Given two such morphisms,

$$(f: X \longrightarrow Y, (\leq_{f^{-1}(y_1)}, \leq_{f^{-1}(y_2)}, \dots, \leq_{f^{-1}(y_n)}))$$

 $(g: Y \longrightarrow Z, (\leq_{f^{-1}(z_1)}, \leq_{f^{-1}(z_2)}, \dots, \leq_{f^{-1}(z_m)})),$

their composite consists of the composite of the underlying finite set maps, together with induced total orders

$$\left(g\circ f:X\longrightarrow Z,\ \left(\leq_{(g\circ f)^{-1}(z_1)},\ \leq_{(g\circ f)^{-1}(z_2)},\ \ldots,\ \leq_{(g\circ f)^{-1}(z_m)}\right)\right),$$

where, given $x, x' \in (g \circ f)^{-1}(z_p)$, we set $x \leq_{(g \circ f)^{-1}(z_p)} x'$ if $f(x) <_{(g)^{-1}(z_p)} f(x')$, or, when f(x) = f(x'), if $x \leq_{f^{-1}(f(x))} x'$.

If C is a category, a C-module (or C-representation) is a functor from C to k-vector spaces for some field k; morphisms of C-modules are natural transformations. We say a C-module V is *finitely generated* if there is a finite list of vectors $v_i \in Vc_i$ so that no proper C-submodule of V contains all the v_i .

Theorem 1. Let M be an orientable, smooth, connected manifold of dimension at least 2 admitting a nowhere-vanishing vector field, and let k be a field. Then, for every i, the assignment $\{1, 2, ..., n\} \mapsto H^i(\operatorname{PConf}^n M, k)$ extends to a finitely generated \mathcal{N} -module over the field k.

Remark 2. On a closed, smooth, connected manifold M, a nowhere-vanishing vector field exists if and only if the Euler characteristic vanishes (see, for example, [AH72, p.552 Satz III]). Odd-dimensional manifolds have vanishing Euler characteristic by Poincaré duality. Thus, any closed, smooth, connected, orientable, odd-dimensional manifold with dim $M \geq 3$ satisfies the conditions of Theorem 1.

Remark 3. We have a forgetful functor $\phi: \mathcal{N} \longrightarrow \mathcal{F}$ to the more familiar category of finite sets that sends a map to its underlying set map. We will see in Theorem 14 that when M is a manifold admitting two vector fields which are everywhere linearly independent, the \mathcal{N} -module structure on the cohomology of the pure configuration spaces of M actually descends to an \mathcal{F} -module structure.

Remark 4. We have another useful functor $\psi: \Delta \longrightarrow \mathcal{N}$ from the category of nonempty finite linear orders and weakly monotone functions. Indeed, any fiber of a monotone function inherits a natural ordering from the domain, and these orderings are compatible. It follows that any \mathcal{N} -representation can be restricted to a Δ -representation (Δ -representations are more commonly known as cosimplicial vector spaces).

Corollary 5. Let M be an orientable, smooth, connected manifold of dimension at least 2 admitting a nowhere-vanishing vector field. Then, for each i, there exists a polynomial P_i such that

(1)
$$\dim H^{i}(\operatorname{PConf}^{n} M, k) = P_{i}(n)$$

for all n > 0. The degree of P_i is at most i if dim $M \ge 3$, and at most 2i if dim M = 2.

Remark 6. That (1) holds for all n sufficiently large relative to i is proved in [CEFN12, Theorem 1.10]. When M is an open manifold, it is shown in [CEF15] that $H^i(\operatorname{PConf}^n M, k)$ carries a different structure, that of an $\mathcal{FI}\#$ -module; it follows that the identity (1) holds for all non-negative n, zero included. So the present result is of interest primarily for closed

manifolds M. In this case, (1) need not hold for n = 0; for instance, when $M = S^3$, we have that

$$\dim H^2(\operatorname{PConf}^n S^3, \mathbb{Q}) = (1/2)(n-1)(n-2)$$

for all n > 0, but obviously not for n = 0. (See e.g. [CT78, Theorem 3].) To see that some hypothesis on M is necessary, note that

$$\dim H^1(PConf^n S^2, \mathbb{Q}) = (1/2)(n^2 - 3n)$$

for all $n \geq 3$, but this is evidently not the case for n = 1, 2.

In fact, the conclusion of Corollary 5 does not require the full \mathcal{N} -module structure on the cohomology of configuration space; it uses only the fact that the sequence of spaces $H^i(\operatorname{PConf}^n M)$ forms a cosimplicial vector space, as follows immediately from the constructions of [BCWW06], as we will explain in Remark 16. However, the extra structure on cohomology allows us to control not only the dimension of the cohomology group, but its structure as a representation of S_n . By a character polynomial, following the notation of [CEF15], we mean a polynomial in the formal variables X_1, X_2, \ldots ; a character polynomial can be interpreted as a character of S_n for every n by taking $X_j(\sigma)$ to be the number of j-cycles in the cycle decomposition of σ . (In particular, X_j is identically 0 on S_n when n < j.) The variable X_j is considered to have degree j.

Corollary 7. Let M be an orientable, smooth, connected manifold of dimension at least 2 admitting a nowhere-vanishing vector field. Then, for each i, there exists a character polynomial P_i such that the character of S_n acting on $H^i(\operatorname{PConf}^n M, \mathbb{Q})$ agrees with P_i for all n > 0. The degree of P_i is at most i if $\dim M \geq 3$, and at most 2i if $\dim M = 2$.

We note that Corollary 5 (for rational coefficients) follows immediately from Corollary 7 by evaluating P_i at $(X_1, X_2, \ldots) = (n, 0, \ldots)$.

As in Remark 6, the novelty of Corollary 5 lies in the uniformity of the bound for closed manifolds M; we already have from [CEF15, Theorem 1.8] that the character of the representation $H^i(\operatorname{PConf}^n M, \mathbb{Q})$ is given by a character polynomial for n large enough relative to i, and for all n when M is open.

The following two corollaries concern rational cohomology of unordered configuration spaces on manifolds with a nonvanishing vector field. The first is a nondecreasing statement for Betti numbers as the number of points grows:

Corollary 8. Let M be a smooth manifold admitting a nowhere-vanishing vector field. Then, unordered configuration space $Conf^n M$ satisfies

$$\dim H^i(\operatorname{Conf}^n M, \mathbb{Q}) \leq \dim H^i(\operatorname{Conf}^{n+1} M, \mathbb{Q}).$$

The next corollary concerns the "replication maps" between configuration spaces defined by Cantero-Palmer [CP14]; this map replaces each of the n points in a configuration with m nearby points spaced out along a short ray in the direction of the nowhere-vanishing vector field. They show that these maps induce isomorphisms in cohomology groups. We provide a new proof of [CP14, Theorem B] in the special case of \mathbb{Q} coefficients.

Corollary 9. Let M be an oriented, connected smooth manifold of dimension at least 2, admitting a nowhere-vanishing vector field. Then, for n large enough relative to i (if

 $\dim M = 2$, when $n \geq 2i$; otherwise when $n \geq i$) and any $m \geq 1$, the replication map $\operatorname{Conf}^n M \to \operatorname{Conf}^{nm} M$ induces an isomorphism

$$\dim H^i(\operatorname{Conf}^{nm} M, \mathbb{Q}) \xrightarrow{\sim} \dim H^i(\operatorname{Conf}^n M, \mathbb{Q}).$$

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1. Representation theory of $\mathcal N$

As we develop the representation theory of \mathcal{N} , we will be able to deduce Corollaries 5, 7, 8, and 9 from Theorem 1. The proof of Theorem 1 appears in §2.

The **simplex category** Δ is the category of finite, non-empty, totally-ordered sets with weakly monotone maps. We use the symbol $[n] = \{1, \ldots, n\}$ to denote the usual ordered set with n elements. Lemma 10 below is an elementary consequence of the classical Dold-Kan correspondence; see [Dol58] or [Wei94, Section 8.4] for a modern treatment.

Write \mathcal{V} for the category of finite dimensional vector spaces over some field k. The Dold-Kan correspondence gives an equivalence of categories between the functor category \mathcal{V}^{Δ} (the category of "cosimplicial vector spaces") and the category of \mathcal{V} -cochain complexes supported in non-negative degree. Under the correspondence, the cochain complex k[-p] consisting of a single 1-dimensional vector space in cohomological degree p becomes an Eilenberg–MacLane representation K(k,p) satisfying dim $K(k,p)[n] = \binom{n-1}{p}$.

Lemma 10 (Dold-Kan Representation Lemma). Let $V: \Delta \longrightarrow \mathcal{V}$ be a Δ -representation over k. The following conditions are equivalent:

- (a) V is finitely generated;
- (b) the sequence $\dim V[n]$ is bounded above by a polynomial in n;
- (c) the cochain complex corresponding to V under Dold-Kan has finite support;
- (d) V has finite length;
- (e) the sequence $\dim V[n]$ coincides with a polynomial in n.

Proof. The cardinality of $\Delta([q], [n])$ (the set of morphisms from [q] to [n] in Δ) is bounded above by the cardinality of $\mathcal{F}([q], [n])$, so a basic projective $k \cdot \Delta([q], -)$ has dimension sequence bounded above by n^q , and $(a) \Longrightarrow (b)$. (The functor $k \cdot \Delta([q], -)$ denotes the composite of the representable functor $[n] \mapsto \Delta([q], [n])$ and the functor sending a finite set to the free k-vector space on that set.) If the cochain complex corresponding to V has k[-p] as a simple constituent, then the dimension sequence of V can have no polynomial bound of degree less than p. It follows that unbounded cochain complexes correspond to Δ -representations with degree sequences that grow faster than any polynomial, and so $(b) \Longrightarrow (c)$. Finitely-supported cochain complexes have finite length, so V must have polynomial dimension sequence; in short, $(c) \Longrightarrow (d) \land (e)$. Since $(d) \Longrightarrow (a)$ and $(e) \Longrightarrow (b)$, we are done.

Proof of Corollary 5 on polynomiality of dimensions. For any $p \in \mathbb{N}$, the dimension sequence of a basic projective \mathcal{N} -representation $k \cdot \mathcal{N}([p], -)$ is bounded above by a polynomial, so the Dold-Kan representation Lemma 10 gives that it is finite length when restricted to Δ . Any finitely generated \mathcal{N} -representation is a quotient of a finite sum of these basic projectives; in particular, the \mathcal{N} -representation $H^i(\mathrm{PConf}^n M, k)$ is such a quotient by Theorem 1. Reapplying Lemma 10 proves Corollary 5. For the bound on degree, we recall from [CEF15, Th 6.3.1, Rem 6.3.3] that $H^i(\mathrm{PConf}^n M, k)$ is bounded above by a polynomial of degree i when $\dim M \geq 3$, or 2i, if $\dim M = 2$. The proof there has a running assumption that k has characteristic 0, but if only an upper bound is desired, that condition is superfluous; the proof uses a filtration of $H^i(\mathrm{PConf}^n M, k)$ by the E^{∞} page of a spectral sequence constructed by Totaro, whose entries are shown in the proof of [CEF15, Th 6.3.1] to have dimension bounded above by a polynomial of degree i (resp. 2i), and this spectral sequence doesn't require rational coefficients.

Theorem 11. An \mathcal{N} -representation is finitely generated if and only if it is finite length.

Proof. We will show that every finitely generated \mathcal{N} -representation restricts to a finite length Δ -representation; since any nontrivial inclusion of \mathcal{N} -representations is in particular a nontrivial inclusion of Δ -representations, the theorem follows. Every finitely generated \mathcal{N} -representation is a quotient of a finite sum of representations of the form $k \cdot \mathcal{N}([p], -)$ for various $p \in \mathbb{N}$, so it suffices to prove the theorem on these basic projectives. As before, the dimension of $k \cdot \mathcal{N}([p], [n])$ is bounded above by a polynomial in n; the claim then follows from the Dold-Kan representation Lemma 10.

Theorem 12. If W is a simple \mathcal{F} -representation, then $W \circ \phi$ is a simple \mathcal{N} -representation. Further, every simple \mathcal{N} -representation arises in this way.

Proof. The first claim is clear, since the functor ϕ is full and surjective on objects. In the other direction, we must show that any two \mathcal{N} -morphisms that coincide after applying ϕ induce the same map under any simple \mathcal{N} -representation.

Given a simple \mathcal{N} -representation V, let m be the smallest integer so that $V[m] \neq 0$, and pick some non-zero vector $v \in V[m]$. Since V is simple, v generates the entire representation. In particular, the vectors (Vf)(v) with $f \in \mathcal{N}([m], [n])$ span V[n]. Given any two morphisms $p, q \in \mathcal{N}([n], [n'])$ such that $\phi(p) = \phi(q)$, we may check that the induced maps Vp and Vq are equal by checking that they act the same way on the spanning vectors (Vf)(v).

Note that $\phi(p \circ f) = \phi(p) \circ \phi(f) = \phi(q) \circ \phi(f) = \phi(q \circ f)$. If $\phi(p \circ f)$ is an injection, then $p \circ f = q \circ f$ since injections in \mathcal{F} have unique lifts to \mathcal{N} . In this case, it follows that $V(p \circ f) = V(q \circ f)$. If $\phi(p \circ f)$ is not an injection, then both maps $V(p \circ f)$ and $V(q \circ f)$ factor through some V[r] with r < m. But V[r] = 0 since m is minimal, and so $V(p \circ f) = 0 = V(q \circ f)$. Thus, in either case $V(p \circ f) = V(q \circ f)$. It follows that V(p)(V(f)(v)) = V(q)(V(f)(v)), and the claim is proved.

Corollary 13. If the field k has characteristic zero, the character of any finitely generated \mathcal{N} -representation V is a polynomial in the cycle-counts X_1, X_2, \ldots for all n > 0.

Proof. By the previous Theorems 11 and 12, it suffices to check the property on the simple \mathcal{F} -representations. Rains [Rai09, Theorem 4.1] deduces this property from work of Putcha [Put96] on the representations of the endomorphism monoid of a finite set. A separate proof appears in [WG14, §6.7.3].

Proof of Corollary 7 on polynomiality of characters. By Theorem 1, the \mathcal{N} -representation $H^i(\operatorname{PConf}^n(M), \mathbb{Q})$ is finitely generated. It follows from Corollary 13 that its character is

given by a polynomial in the cycle-counts. The bound on degree is proved as in the proof of Corollary 5.

Proof of Corollaries 8 and 9 on cohomology of unordered configuration spaces. Working with rational coefficients, $H^i(\operatorname{Conf}^n M, \mathbb{Q}) = H^i(\operatorname{PConf}^n M, \mathbb{Q})^{S_n}$. Following the notation of [WG14], the only simple N-representations in which trivial S_n -representations appear are those denoted $D_0, D_1, C_2, C_3, C_4, \ldots$, which we now describe explicitly.

A basis for $C_k[n]$ is given by all k-element subsets $S \subseteq [n]$. An \mathcal{N} -arrow $f:[n] \longrightarrow [n']$ acts by $S \mapsto \phi(f)(S)$, provided $\phi(f)(S)$ still has k elements; if $\phi(f)(S)$ has fewer than k elements, then $S \mapsto 0$. In particular, dim $C_k^{S_n}$ is 1 when $n \geq k$ and 0 when n < k. The simple D_1 is onedimensional at every object except [0], with arrows acting by the one-by-one matrix (1); D_0 is one-dimensional at [0] and zero everywhere else. From this description, we see that every simple \mathcal{N} -representation V defined over \mathbb{Q} satisfies $0 \leq \dim V[n]^{S_n} \leq \dim V[n+1]^{S_{n+1}} \leq 1$ for $n \geq 1$, and so dim $H^i(\operatorname{Conf}^n M, \mathbb{Q})^{S_n} \leq \dim H^i(\operatorname{Conf}^{n+1} M, \mathbb{Q})^{S_{n+1}}$. This proves Corollary 8.

In order to deduce the claim about replication maps, note that the simples appearing in the \mathcal{N} -representation $H^i(\operatorname{Conf}^n M, \mathbb{Q})$ cannot grow faster than a polynomial of degree d=2i(d=i) if the manifold has dimension at least 3) by the bounds on $\dim H^i(\operatorname{Conf}^n M, \mathbb{Q})$ given in [CEF15, Theorem 6.3.1 and Remark 6.3.3]. Since $n \mapsto \dim C_k[n]$, $n \ge 1$, is a polynomial of degree k, we are only concerned with the simples $D_1, C_2, C_3, \ldots, C_d$.

Let V be one of these simples. Given a finite set map $f:[n] \longrightarrow [n']$ whose image has size at least $d \geq 1$, we show that the composite $\overline{f}: V[n]^{S_n} \longrightarrow V[n] \xrightarrow{f_*} V[n'] \longrightarrow V[n']_{S_{n'}} \xrightarrow{\sim}$ $V[n']^{S_{n'}}$ is an isomorphism. The case of $V=D_1$ is immediate. If $V=C_k$ with $k \leq d$, then pick a k-element subset of [n] so that f restricts to an injection on these elements. It follows that this subset maps via f_* to a nonzero vector, and that \overline{f} maps the average of all k-element subsets to a nonzero vector. Recalling that the source and target of \overline{f} are at most one-dimensional, we are done.

It follows that the map \overline{f} is an isomorphism for any simple whose dimension grows as a polynomial of degree at most d, and hence for any finite extension of such simples, by the 5-lemma. The claim follows for the finite length N-representation $H^i(\operatorname{Conf}^n M, \mathbb{Q})$. This proves Corollary 9.

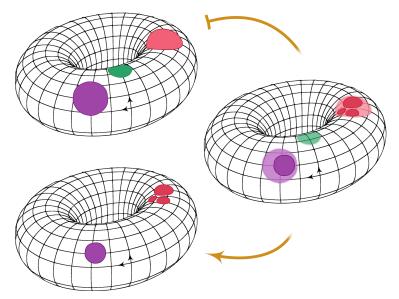
2. Proof of Theorem 1

Let M be a Riemannian manifold equipped with l everywhere linearly independent vector fields, and write $\tau \subseteq TM$ for the resulting trivial vector subbundle of the tangent bundle of M. For V an inner product space, let $B(V,\varepsilon) = \{v \in V \text{ such that } |v| < \varepsilon\}$. An embedding $\iota: B(\mathbb{R}^l, 1) \longrightarrow M$ is an exp-embedding if there is some $\varepsilon > 0$ so that ι coincides with scaling by ε followed by the exponential map $B(\tau_{\iota(0)},\varepsilon) \longrightarrow M$. Write $\operatorname{eEmb}(B(\mathbb{R}^l,1),M)$ for the space of exp-embeddings. We also extend the definition of exp-embedding to include embeddings of disjoint unions of balls that restrict to exp-embeddings on every component.

The disjoint union of n standard l-balls $B_{\sqcup n}^l = \sqcup_{i=1}^n B(\mathbb{R}^l, 1)$ has a natural choice of leverywhere linearly independent vector fields. Evidently, the exp-embeddings between these spaces are closed under composition; write $\mathcal{F}E_l$ for the resulting category with objects [n]for $n \in \mathbb{N}$ and $\operatorname{Hom}_{\mathcal{F}E_l}([n],[n']) = \operatorname{eEmb}(B^l_{\sqcup n},B^l_{\sqcup n'})$. Contracting exp-embedded balls to their centers provides a homotopy equivalence

$$\operatorname{eEmb}(B_{\square n}^l, M) \simeq \operatorname{PConf}^n(M),$$

so $\operatorname{Emb}(B^l_{\sqcup n}, M)$ is a reasonable model for configuration space. The category $\mathcal{F}E^{op}_l$ acts on $\operatorname{eEmb}(B^l_{\sqcup n}, M)$ by precomposition. We illustrate precomposition with an exp-embedding of four disjoint disks into three disjoint disks in the case $M = S^1 \times S^1$:



It follows that $[n] \mapsto H^i(\operatorname{eEmb}(B^l_{\sqcup n}, M)) \simeq H^i(\operatorname{PConf}^n(M))$ is an $\mathcal{F}E_l$ -representation. Moreover, since homotopic maps induce the same linear map on cohomology, this action descends to $\pi_0 \mathcal{F}E_l$, the quotient category induced by taking path components of every homspace. We have mostly proved the following theorem, which is stronger than Theorem 1.

Theorem 14. Let M be a smooth manifold admitting l everywhere linearly independent vector fields, and let k be a coefficient field. The assignment $[n] \mapsto H^i(\operatorname{PConf}^n(M); k)$ is a representation of $\pi_0 \mathcal{F} E_l$, a category that simplifies further depending on l:

$$\pi_0 \mathcal{F} E_l = \begin{cases} \mathcal{F} \mathcal{I} & : l = 0 \\ \mathcal{N} & : l = 1 \\ \mathcal{F} & : l \ge 2. \end{cases}$$

Further, if M is connected, orientable, and dimension at least two, then this representation is finitely generated.

Proof. The construction explained above gives the action of $\pi_0 \mathcal{F} E_l$; we must show that these representations are finitely generated. The paper [CEF15] deduces finite generation even for the case l=0 from the spectral sequence of Totaro [Tot96], from which it follows for every other l (since $\pi_0 \mathcal{F} E_0$ is naturally a subcategory of every other $\pi_0 \mathcal{F} E_l$.)

The simplification of the category $\pi_0 \mathcal{F} E_l$ depends on the following easy observations. When l=0, exp-embeddings of 0-balls are just injections among disjoint unions of points; it follows that $\pi_0 \mathcal{F} E_0 \simeq \mathcal{F} \mathcal{I}$, the category of finite sets with injections. For l=1, a configuration of line segments in a line segment (up to homotopy) is the same as an ordering of the segments; this provides the isomorphism $\pi_0 \mathcal{F} E_1 \simeq \mathcal{N}$. When $l \geq 2$, eEmb $(B_{\sqcup n}^l, B^l)$ is a connected space and so $\pi_0 \mathcal{F} E_l = \mathcal{F}$.

Remark 15. The notation $\mathcal{F}E_l$ is meant to evoke the "category of operations" for the little l-balls operad E_l , introduced by May-Thomason in [MT78]. (Another popular notation for this category is E_l^{\otimes} .) Our category is essentially an unbased analog of their category.

Remark 16. A version of this construction for l=1 is essentially present in work of Berrick, Cohen, Wong, and Wu [BCWW06], [Wu10], although they are content to use the subcategory $\Delta \subseteq \mathcal{N}$. It must be noted that Corollary 5, the statement on dimensions of cohomology of configuration spaces of manifolds with flows, could be deduced immediately from [BCWW06] and the Dold-Kan representation Lemma 10. Nevertheless, such a proof seems not to be present in the literature, despite considerable interest in the asymptotic behavior of these dimension sequences; and the use of \mathcal{N} as opposed to Δ is necessary if one wishes to obtain stability theorems for the character of the symmetric group actions, as in Corollary 7.

Question 17. Is it possible to remove the assumption of orientabilty from Theorem 14? We have included the assumption in order to deduce finite generation from prior work. In principle, however, the Leray spectral sequence exposited by Totaro [Tot96] should accommodate non-orientable manifolds. Can an equally explicit description of E_2 be obtained in this case?

REFERENCES

- [AH72] Paul Alexandroff and Heinz Hopf. *Topologie. Band I.* Chelsea Publishing Co., Bronx, N. Y., first edition, 1972. Grundbegriffe der mengentheoretischen Topologie, Topologie der Komplexe, topologische Invarianzsätze und anschliessende Begriffsbildungen. Verschlingungen im *n*-dimensionalen euklidischen Raum, stetige abbildungen von Polyedern, Including an addendum based on an article by Hing Tong, George Kozlowski and Mary Powderly ("On a problem of Alexandroff and Hopf", Bull. Inst. Math. Acad. Sinica 3 (1975), no. 1, 15–16).
- [Arn69] V.I. Arnol'd. The cohomology ring of the colored braid group. Mathematical notes of the Academy of Sciences of the USSR, 5(2):138–140, 1969.
- [BCWW06] A. J. Berrick, F. R. Cohen, Y. L. Wong, and J. Wu. Configurations, braids, and homotopy groups. J. Amer. Math. Soc., 19(2):265–326, 2006.
- [CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb. FI-modules and stability for representations of symmetric groups. *Duke Math. J.*, 164(9):1833–1910, 2015.
- [CEFN12] Thomas Church, Jordan S. Ellenberg, Benson Farb, and Rohit Nagpal. FI-modules over noe-therian rings, 2012. To appear in Geom. Topol. arXiv:1210.1854.
- [CF13] Thomas Church and Benson Farb. Representation theory and homological stability. *Adv. Math.*, 245:250–314, 2013.
- [Chu12] Thomas Church. Homological stability for configuration spaces of manifolds. *Invent. Math.*, 188(2):465–504, 2012.
- [Coh88] F. R. Cohen. Artin's braid groups, classical homotopy theory, and sundry other curiosities. In Braids (Santa Cruz, CA, 1986), volume 78 of Contemp. Math., pages 167–206. Amer. Math. Soc., Providence, RI, 1988.
- [CP14] Federico Cantero and Martin Palmer. On homological stability for configuration spaces on closed background manifolds, 2014. arXiv:1406.4916.
- [CT78] F. R. Cohen and L. R. Taylor. Configuration spaces: applications to Gelfand-Fuks cohomology. Bull. Amer. Math. Soc., 84(1):134–136, 1978.
- [Dol58] Albrecht Dold. Homology of symmetric products and other functors of complexes. Ann. of Math. (2), 68:54–80, 1958.
- [Knu14] Ben Knudsen. Betti numbers and stability for configuration spaces via factorization homology, 2014. arXiv:1405.6696.
- [McD75] Dusa McDuff. Configuration spaces of positive and negative particles. Topology, 14:91–107, 1975.
- [MT78] J. P. May and R. Thomason. The uniqueness of infinite loop space machines. *Topology*, 17(3):205–224, 1978.
- [PR02] T. Pirashvili and B. Richter. Hochschild and cyclic homology via functor homology. K-Theory, $25(1):39-49,\ 2002.$
- [Put96] Mohan S. Putcha. Complex representations of finite monoids. *Proc. London Math. Soc.* (3), 73(3):623–641, 1996.
- [Rai09] Eric M. Rains. The action of S_n on the cohomology of $\overline{M}_{0,n}(\mathbb{R})$. Selecta Math. (N.S.), 15(1):171–188, 2009.

ALGEBRAIC STRUCTURES ON COHOMOLOGY OF CONFIGURATION SPACES OF MANIFOLDS WITH FLOWS

[RW13]	Oscar Randal-Williams.	Homological	stability fo	or unordered	configuration sp	aces. $Q. J.$. Math.,
	64(1):303–326, 2013.						

- [Seg74] Graeme Segal. Categories and cohomology theories. Topology, 13:293–312, 1974.
- [Tot96] Burt Totaro. Configuration spaces of algebraic varieties. Topology, 35(4):1057–1067, 1996.
- [Wei94] Charles A. Weibel. An introduction to homological algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.
- [WG14] John D. Wiltshire-Gordon. Uniformly presented vector spaces, 2014. arXiv:1406.0786.
- [Wu10] Jie Wu. Simplicial objects and homotopy groups. In Braids, volume 19 of Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., pages 31–181. World Sci. Publ., Hackensack, NJ, 2010.

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